Noise-induced phase synchronization enhanced by dichotomic noise

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We study the nonlinear response of a stochastic bistable system driven by both a weak periodic signal and a dichotomic noise in terms of stochastic phase synchronization. We show that the effect of noise-induced phase synchronization can be significantly enhanced by the addition of a dichotomic noise.

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I. INTRODUCTION

One of the surprising phenomena that show the constructive role played by noise in nonlinear systems is stochastic resonance (SR) [1]. It can be observed in the nonmonotonic dependence of the signal-to-noise ratio (SNR) and the spectral power amplification (SPA) on the noise intensity. SR has been thoroughly investigated during the last ten years (for a comprehensive review, see [2]).

An effect that is closely connected with SR is the phenomenon of noise-induced synchronization which is observed in the locking of the output phase to the phase of the input signal [3–5]. Two cases should be distinguished here: first, where the phase of the output follows the input phase on the average but large fluctuations can occur, and second, the case where, in addition to the first requirement, the fluctuations of the input-output phase difference are very small. The first case we call frequency locking, the second the phase locking effect. In the simplest case frequency locking can be detected by a plateau of the mean frequency of the output switching where it attains values close to the frequency of the input signal. For the second case (phase locking) it is possible to construct an effective diffusion coefficient as a measure of the fluctuations of the input-output phase difference, which in the region of frequency locking attains small values.

We note that synchronization, being a nonlinear phenomenon, cannot be described in terms of linear response theory (LRT) [6]. This makes an important difference between conventional SR and noise-induced phase synchronization: while SR can be observed for very weak signals and can be described by LRT, noise-induced phase synchronization requires a significant amplitude of the input [7].

In this paper we study the generic two-state model of SR [8,9] and analyze the influence of an additional dichotomic Markovian process which is uncorrelated to the periodic input.The effect of this additional dichotomic noise on SR in the context of spectral measures was reported recently in [10]. There, within LRT, an enhancement of both the SPA and SNR with increasing amplitude of the dichotomic noise was shown. In the following we will prove that beyond LRT an enhancement also occurs for stochastic phase synchronization and that synchronization can be observed for smaller amplitudes of the signal in comparison with the case without dichotomic noise.

The paper is organized as follows. In Sec. II we sketch the bistable system subjected to an additional dichotomic noise. In Sec. III the phase description is introduced for the above system driven by an external periodic signal. Simulation results for a harmonic input signal are presented in Sec. IV whereas analytic calculations are performed in Sec. V for a discrete periodic input signal. Conclusions are given in Sec. VI.

II. A BISTABLE MODEL WITH DICHOTOMIC NOISE

One of the canonical models for studying SR is the driven overdamped bistable oscillator [2]. Here, we consider a situation where this system is influenced by an additional stochastic switching process which is independent of the driving input signal. Such a combination of processes can occur in the context of different scenarios: (1) the dichotomic Markovian dynamics can be superimposed onto a periodic input signal; (2) it can be an externally applied control [11]; or (3) it can be related to some internal degree of freedom. Regardless of the specific scenario, the response of the system to the periodic signal is modified because of a modulation of the switching rates [2,4,5,12,13]. Consequently we will investigate how modifications of the effective rates change the response of the system to an external periodic driving.

In general, we assume the following situation. A bistable potential with Gaussian white noise is subjected to (1) an additive dichotomic Markovian process with zero mean which randomly modulates the potential shape, and (2) a periodic external input signal. Both processes modify the noise-dependent time scales and give rise to SR and stochastic phase synchronization.

Mapping of the bistable system onto two states [9] captures those features of the continuous system that are essential for SR. The two-state model which after inclusion of the dichotomic Markovian noise yields a four-state model is sketched in Fig. 1.

The left and right states correspond to positions in the left and right wells which, in the absence of the dichotomic Markovian noise ($B=0$), are separated by a barrier of height ΔU. Thermal noise will induce stochastic transitions between the two wells of the bistable potential. In the limit of small noise intensity, which means $D \ll \Delta U$, the rate of escape from one of the two symmetric wells is given by $a(D)=a_0 \exp(-\Delta U/D)$ [14]. Throughout the article we fix dimensionless $\Delta U=0.25$. The prefactor a_0 sets an upper bound on the noise-dependent rates. Since no other process

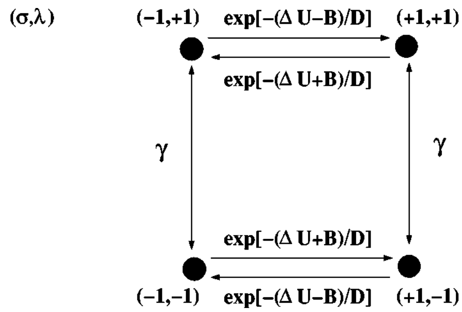

FIG. 1. The basic four-state model.

should be faster, all other rates occurring in our setup have to be (much) smaller. By proper rescaling of time we can always set $a_0=1$.

In the following, by γ we denote the switching rate of the dichotomic noise. For $B>0$ this process modifies the transition rates of the stochastic two-state dynamics: the single rate of escape from one of the two symmetric wells splits into two values

$$a_1=\exp\left(-\frac{\Delta U+B}{D}\right),\quad a_2=\exp\left(-\frac{\Delta U-B}{D}\right). \tag{1}$$

By $B<\Delta U$ we denote the amplitude of the dichotomic noise while its instantaneous value is written as $\lambda(t)\in\{-1,1\}$. The two states of the output of the bistable dynamics are labeled by $\sigma(t)\in\{-1,1\}$. With these definitions the time dependent modified rates can be expressed by

$$W^0(\sigma,\lambda)=\exp\left(-\frac{\Delta U+\sigma\lambda B}{D}\right)$$
$$=\frac{1}{2}[(a_1+a_2)-\sigma\lambda(a_2-a_1)]. \tag{2}$$

In the following, let $p(\sigma,\lambda)$ be the probability of the instantaneous configuration (σ,λ). The master equation for the stochastic dynamics reads

$$\frac{d}{dt}p(\sigma,\lambda)=W^0(-\sigma,\lambda)p(-\sigma,\lambda)-W^0(\sigma,\lambda)p(\sigma,\lambda)$$
$$+\gamma[p(\sigma,-\lambda)-p(\sigma,\lambda)]. \tag{3}$$

This master equation was used in [4,5] to obtain the cross-correlation function between the output and the dichotomic noise $\langle\sigma\lambda\rangle$. In the asymptotic limit this correlator exhibits a nonmonotonic dependence on the thermal noise intensity D. Thus, it evidences maximal correlations between the output and the dichotomic noise for an optimal value of noise intensity D—the distinguishing feature of SR.

III. PHASE DESCRIPTION OF THE BASIC MODEL

In the present paper the evolution of the system presented above will be considered in terms of stochastic phase dynamics. To introduce a phase description for the model addressed in the previous section additionally driven by a periodic input signal, we have to define three different phases: one for the output ϕ_σ, one for the dichotomic noise ϕ_λ, and one for the input signal ϕ_d.

Let us assume that the output switching events occur at times t_k with $k=0,1,2,\ldots$. Then, for this point process, the instantaneous phase of the output $\phi_\sigma(t)$ can be defined as

$$\phi_\sigma(t)=\pi\sum_k \theta(t-t_k). \tag{4}$$

As a result, switching events are accompanied by accumulating jumps of the phase, i.e., each time the system switches between the left and the right state the phase changes by π. Obviously, this leads to

$$\cos[\phi_\sigma(t)]=\sigma(t). \tag{5}$$

Using an analogous definition for the dichotomic noise $\lambda(t)$ yields

$$\phi_\lambda(t)=\pi\sum_j \theta(t-t_j),\quad \cos[\phi_\lambda(t)]=\lambda(t), \tag{6}$$

where t_j are now the switching times of the dichotomic noise.

Two kinds of input signal will be considered: (1) The first is a harmonic continuous signal $d(t)=A\cos(\Omega t-\theta_0)$ with instantaneous phase $\phi_d(t)=\Omega t-\theta_0$; this will be used in our numerical simulation. (2) The second is a periodic discrete input $d(t)=\mathrm{sgn}[\cos(\Omega t-\theta_0)]$ with an instantaneous phase given by

$$\phi_d(t)=\pi\sum_n \theta(t-t_n),\quad \cos[\phi_d(t)]=d(t) \tag{7}$$

with deterministic switching times $t_n=(n\pi+\theta_0)/\Omega$. Our analytic approach will be based on this discrete variant.

Since we aim at a description of the effective phase synchronization we focus our attention on the instantaneous phase difference between the output and the periodic input,

$$\varphi=\phi_\sigma-\phi_d. \tag{8}$$

The time derivative of its average gives the difference between the mean frequencies of the output and the input jumps,

$$\langle\omega\rangle=\frac{d}{dt}\langle\varphi\rangle=\langle\omega_\sigma\rangle-\langle\omega_d\rangle. \tag{9}$$

Here $\langle\omega_\sigma\rangle$ is the mean frequency of the output switchings and in the following will be called the mean switching frequency (MSF). In general, the quantities occurring in Eq. (9) are time dependent for two reasons. The periodic input introduces a nonstationary (sometimes called cyclostationary) aspect, which, however, can be absorbed by initial phase averaging; we will come back to this point later. Secondly, and aside from the periodic driving, the system has to relax to some asymptotic stationary value. This aspect was first addressed in detail in [5] and again will show up in kinetic equations for the correlators [cf. Eqs. (16) and (17) below]. In the following we will always focus on asymptotic stationary quantities measured after phase averaging and after relevant relaxation times have passed.

Let us first repeat the general conditions for forced synchronization. The synchronization of an output with an input reveals itself by two effects: frequency and phase locking. The first effect generally means that the ratio of the mean switching frequency of the output and the mean input frequency forms a rational number, i.e.,

$$\frac{\langle \omega_\sigma \rangle}{\langle \omega_d \rangle} = \frac{m}{n}. \qquad (10)$$

In the following, we will always restrict our analysis to the case $m=n=1$ and constant input frequency $\langle \omega_d \rangle = \Omega$ and calculate the output MSF in the presence of both processes—the input signal and the dichotomic noise. The MSF will measure how the dichotomic noise alters the frequency synchronization between the output and the input signals.

Additionally, we look for phase locking which, in the case $m=n=1$ and without noise, imposes the following constraint on the phase difference:

$$|\varphi| < \text{const}, \qquad (11)$$

which is valid for all times. In the presence of noise condition (11) does not hold true rigorously. However, under appropriate conditions (sufficiently small or optimal noise) for a long period τ the phase difference φ varies only slightly, i.e.,

$$|\varphi(t) - \varphi(t_0)| \ll O(\pi) \quad \text{for} \quad t_0 < t < t_0 + \tau; \qquad (12)$$

the system experiences a locking episode of duration τ. Locking episodes are interrupted by rare fluctuations which cause a phase slip, i.e., the phase difference φ changes by an order of π, after which another locking episode starts. In cases when $\langle \tau \rangle$ is large compared to the period of the external force one may speak about effective phase locking.

Frequency locking without phase locking can occur since it is possible to obey $\langle \omega \rangle \approx 0$ and still experience large phase difference fluctuations—where it varies not by drift but by diffusion. The converse is not true, i.e., effective phase locking always implies frequency locking since it requires both vanishing diffusion and vanishing drift. This reveals that generally phase locking is a stronger effect than frequency locking. In [4,5] the regions of frequency and phase locking roughly coincided.

The motion of the phase difference φ is quantified by a related effective diffusion coefficient

$$\mathcal{D}_{\text{eff}} = \frac{1}{2}\frac{d}{dt}\langle(\varphi - \langle\varphi\rangle)^2\rangle = \frac{1}{2}\frac{d}{dt}(\langle\varphi^2\rangle - \langle\varphi\rangle^2) \qquad (13)$$

which in the asymptotic limit $t \to \infty$ approaches a constant.

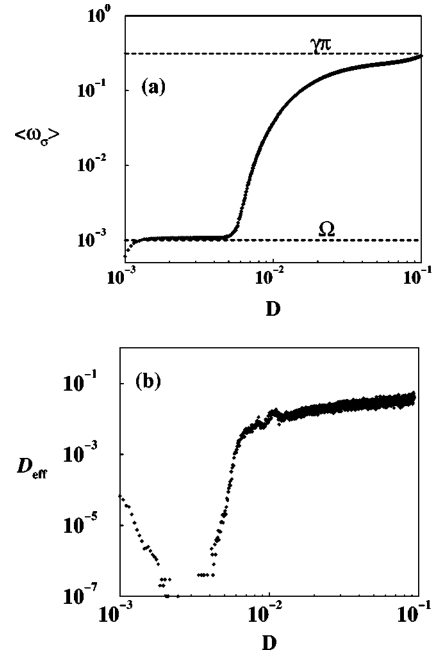

FIG. 2. Ensemble simulation of the bistable system with a harmonic input signal ($A=0.03$ and $\Omega=0.001$) and with a dichotomic process ($B=0.215$ and $\gamma=0.1$). Mean output switching frequency $\langle \omega_\sigma \rangle$ (a) and effective diffusion coefficient $\mathcal{D}_{\text{eff}}$ (b).

IV. SIMULATION RESULTS

To support the arguments to be used in our analytic approach we first present the results of numerical simulations. The averages defining 1:1 frequency (10) and phase (13) locking were computed using ensembles of trajectories. Each single realization was computed in a standard fashion employing instantaneous transition rates (see, e.g., [9], Sec. IV). The harmonic input signal was weak, i.e., $A=0.03 \ll B=0.215$, and slow, $\Omega=0.001 \ll \gamma=0.1$ (with γ being the switching rate of the dichotomic Markovian process). The time step was chosen as $\Delta t=10^{-2}$ s and the simulation time was $5\times10^6\Delta t$. For each selected noise intensity D100 realizations were combined, thus forming the ensemble.

Our results are shown in Fig. 2. As indicated by the first plateau for small noise intensities the MSF of the output is locked to the frequency of the periodic input signal Ω. In the same region of D the diffusion coefficient possesses a minimum. Both effects together evidence effective phase locking. It is the very region of noise intensity where the SPA for the two-state system with dichotomic noise achieves the first strong maximum [10]. We underline that, contrary to the case without dichotomic noise, this effective phase synchronization already occurs for rather small input amplitudes A. A second plateau occurs for higher noise intensities around the frequency of the dichotomic noise $\pi\gamma$ without being accompanied by a decrease of $\mathcal{D}_{\text{eff}}$. The latter effect is a mere consequence of the fact that the effective diffusion coefficient is defined with respect to the phase difference $\varphi=\phi_\sigma-\phi_d$.

V. ANALYTIC APPROACH

An analytic approach to the synchronization effect is feasible in the case of a dichotomic periodic input. Let

$p(\sigma,\lambda,d)$ denote the probability of observing the output σ, the state of the dichotomic noise λ, and the input d at time t (conditioned by some initial configuration at time t_0). Then the master equation reads

$$\frac{d}{dt}p(\sigma,\lambda,d)=W(-\sigma,\lambda,d)p(-\sigma,\lambda,d) -W(\sigma,\lambda,d)p(\sigma,\lambda,d)+\gamma[p(\sigma,-\lambda,d) -p(\sigma,\lambda,d)]+\sum_{n=0}^{\infty}\delta\left(t-\frac{n\pi+\theta_0}{\Omega}\right) \times[p(\sigma,\lambda,-d)-p(\sigma,\lambda,d)] \qquad (14)$$

where $W(\sigma,\lambda,d)$ is given by the formula

$$W(\sigma,\lambda,d)=W^0(\sigma,\lambda)\exp\left(-\frac{A}{D}\sigma d\right) =\frac{1}{2}[(a_1+a_2)-\sigma\lambda(a_2-a_1)] \times\left[\cosh\left(\frac{A}{D}\right)-\sigma d\sinh\left(\frac{A}{D}\right)\right]. \qquad (15)$$

For the dichotomic periodic input no linearization with respect to the amplitude of the signal has to be performed. This allows for calculations beyond the regime of LRT, i.e., for any value of the signal amplitude $A<\Delta U-B$ (subthreshold condition).

The classical description of SR in the case of discrete input signal is based on the cross-correlation functions between output and input signals, $\langle\sigma d\rangle$. The kinetic equations for this function as well as for the cross correlator $\langle\sigma\lambda\rangle$ between output and dichotomic noise can be obtained directly from the master equation (14):

$$\frac{d}{dt}\langle\sigma\lambda\rangle=-\left[2\gamma+(a_1+a_2)\cosh\left(\frac{A}{D}\right)\right] \times\langle\sigma\lambda\rangle-(a_2-a_1) \times\sinh\left(\frac{A}{D}\right)\langle\sigma d\rangle+(a_2-a_1)\cosh\left(\frac{A}{D}\right), \qquad (16)$$

$$\frac{d}{dt}\langle\sigma d\rangle=-\left[2\frac{\Omega}{\pi}+(a_1+a_2)\cosh\left(\frac{A}{D}\right)\right] \times\langle\sigma d\rangle-(a_2-a_1) \times\sinh\left(\frac{A}{D}\right)\langle\sigma\lambda\rangle+(a_2+a_1)\sinh\left(\frac{A}{D}\right). \qquad (17)$$

Let us denote the time independent stationary values by the superscript s. Both of the functions $\langle\sigma d\rangle^s$ and $\langle\sigma\lambda\rangle^s$ reveal a nonmonotonic dependence on the thermal noise intensity D which is the distinguishing feature of stochastic resonance (see panel (b) of Fig. 3). The behavior of the cross correlator $\langle\sigma d\rangle^s$ is qualitatively the same as the behavior of the SPA discussed in [10].

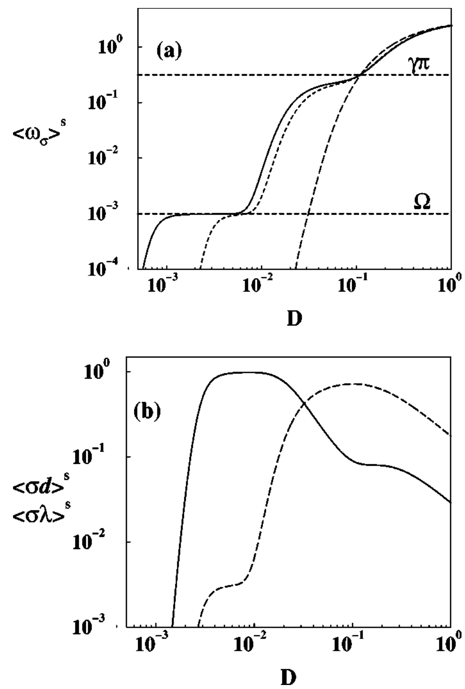

FIG. 3. Output MSF (a) as a function of D for various B, 0 (long dashed), 0.2 (dashed), 0.215 (solid), in comparison with the cross correlators (b) $\langle\sigma d\rangle^s$ (solid) and $\langle\sigma\lambda\rangle^s$ (dashed) for $B=0.2$. Other parameters: $\Omega=0.001$, $A=0.03$, $\gamma=0.1$.

A. Frequency locking

To describe the frequency locking effect analytically the MSF in the presence of both dichotomic noise and the external periodic signal has to be calculated. The rates (15) rewritten in the phase description introduced in Sec. III read

$$W(\phi_\sigma,\phi_\lambda,\phi_d)=a(D)\exp\left(-\frac{B}{D}\cos(\phi_\sigma-\phi_\lambda)\right) \times\exp\left(-\frac{A}{D}\cos(\phi_\sigma-\phi_d)\right). \qquad (18)$$

The master equation for the evolution of $p(\phi_\sigma,\phi_\lambda,\phi_d)$ is achieved by reformulating Eq. (14) employing the rates (18), which yields

$$\frac{d}{dt}p(\phi_\sigma,\phi_\lambda,\phi_d) =(F_\sigma-1)W(\phi_\sigma,\phi_\lambda,\phi_d)p(\phi_\sigma,\phi_\lambda,\phi_d) +\gamma(F_\lambda-1)p(\phi_\sigma,\phi_\lambda,\phi_d) +\sum_{n=0}^{+\infty}\delta\left(t-\frac{n\pi+\theta_0}{\Omega}\right)(F_d-1)p(\phi_\sigma,\phi_\lambda,\phi_d) \qquad (19)$$

where $F_\sigma f(\phi_\sigma,\phi_\lambda,\phi_d)=f(\phi_\sigma-\pi,\phi_\lambda,\phi_d)$ and analogous definitions apply to F_λ and F_σ.

In Eq. (19) two dichotomic processes enter: the periodic input signal and the Markovian dichotomic noise. The difference between them lies in the growth of an ensemble related initial variance: for the periodic signal an ensemble prepared with identical initial phase θ_0 will switch uniformly at times $t_n=(n\pi+\theta_0)\Omega^{-1}$; hence, even ensemble averages change discontinuously and the zero initial variance never grows. This is different for the dichotomic Markovian noise where jumps occur at different random moments. Hence, ensemble averages are continuous functions of time and even a vanishing initial variance will grow as time elapses. A difference can also be seen in the transition rate which, in the case of the periodic signal, is time dependent and which is the time independent rate γ for the Markovian dynamics.

To remove the nonstationary (cyclostationary) aspect from ensemble averages one can perform an additional average over the initial phase using the assumption of a uniform distribution, i.e., $P(\theta_0)=1/2\pi$.

From Eq. (19) the evolution equation for the average phase difference is obtained, which, after averaging over the initial phase θ_0 of the input, reads

$$\frac{d}{dt}\langle\varphi\rangle=\pi\langle W(\phi_\sigma,\phi_\lambda,\phi_d)\rangle-\Omega. \tag{20}$$

By definition we identify the MSF of the output calculated with both the dichotomic noise and the periodic input signal as

$$\langle\omega_\sigma\rangle=\pi\langle W(\phi_\sigma,\phi_\lambda,\phi_d)\rangle. \tag{21}$$

The cross-correlation functions between output and input signal and output and dichotomic noise are redefined in the phase description as $\langle\sigma d\rangle=\langle\cos\varphi\rangle$ and $\langle\sigma\lambda\rangle=\langle\cos(\phi_\sigma-\phi_\lambda)\rangle$. This can be checked easily using trigonometric identities as well as definitions from Sec. III. The MSF expressed as a function of these cross correlators reads

$$\frac{2}{\pi}\langle\omega_\sigma\rangle^s=(a_1+a_2)\cosh\left(\frac{A}{D}\right)-(a_2-a_1)\langle\cos(\phi_\sigma-\phi_\lambda)\rangle^s \times\cosh\left(\frac{A}{D}\right)-(a_1+a_2)\langle\cos\varphi\rangle^s\sinh\left(\frac{A}{D}\right). \tag{22}$$

Here, we also use the fact that the input signal and the dichotomic noise are uncorrelated $\langle\cos(\phi_\lambda-\phi_d)\rangle^s=0$. Note that a plateau of the stationary MSF can occur only when either of the two stationary correlators gains sufficient weight, be it $\langle\cos\varphi\rangle^s$ giving rise to a plateau at low noise intensities, or $\langle\cos(\phi_\sigma-\phi_\lambda)\rangle^s$ which slows down the growth of the MSF at larger noise intensities.

The result of our analytic treatment is visualized in Fig. 3. In panel (a), the output MSF is presented for a weak and slow signal ($A=0.03,\Omega=0.001$) and a fast dichotomic noise ($\gamma=0.1$) and for three different amplitudes B. For sufficiently large B two distinct regions of synchronization can be seen. The plateau occurring for small noise intensities corresponds to a region where the MSF is locked to the frequency Ω of the periodic input. Due to our definition of the phase φ it corresponds to the regime of noise-induced frequency synchronization. As can be seen from panel (b) this locking region is closely connected to the broad flat peak of the input-output cross correlator which, for $B=0.2$, attains a value near unity.

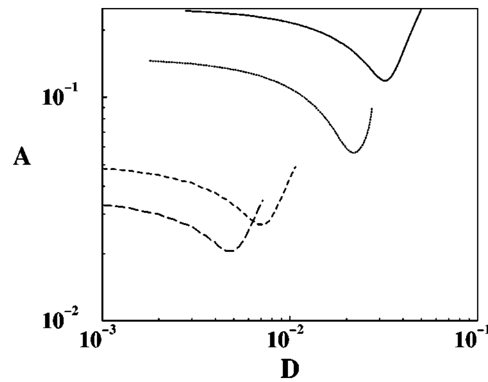

FIG. 4. Arnold-like tongues calculated for $\Omega=0.001$ and $\gamma=0.1$, and four values of $B=0$ (solid), 0.1 (dotted), 0.2 (dashed), and 0.215 (long dashed).

We want to emphasize that the addition of a fast switching dichotomic noise effects the emergence of a locking regime which is never found for the bistable dynamics driven by a *weak* periodic input signal alone. Previous investigations [5] (corresponding to $B=0$) have shown this kind of forced synchronization only for rather large amplitudes A. Hence, the addition of a dichotomic noise, realized in practice, for example, through an external driving, should improve the coherence between the response and the periodic input even for rather small amplitudes of the input signal.

In Fig. 4 we present Arnold-like tongues calculated for different values of the amplitude of the dichotomic noise B. As is seen, with increasing amplitude B the amplitude A necessary to obtain a plateau of $\langle\omega_\sigma\rangle$ is significantly lowered. This clearly illustrates the synchronization enhancing role of the additional dichotomic noise. With increasing B optimal noise shifts to smaller values and the minimal A necessary for frequency synchronization decreases rapidly (note the logarithmic scale in Fig. 4).

More insight into the beneficial role of dichotomic noise is gained by inspection of Fig. 5. Here, we compare the output MSF with the mean frequency of a periodically driven

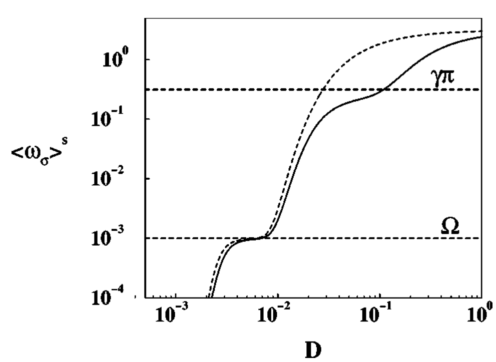

FIG. 5. Comparison of the output MSF for $\gamma=0.1, B=0.2$ (solid) with a periodically driven two-state system with effective barrier $\Delta U_{\rm eff}=0.05$ (dashed). Other parameters: $A=0.03$, $\Omega=0.001$.

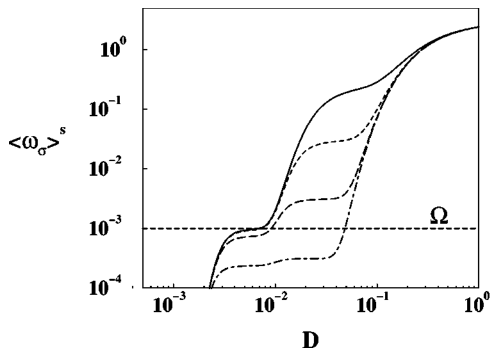

FIG. 6. Output MSF as a function of D for $B=0.2$ and four values of $\gamma=0.1$ (solid), 0.01 (dashed), 0.001 (long dashed), and 0.0001 (dot dashed). Other parameters: $\Omega=0.001$ and $A=0.03$.

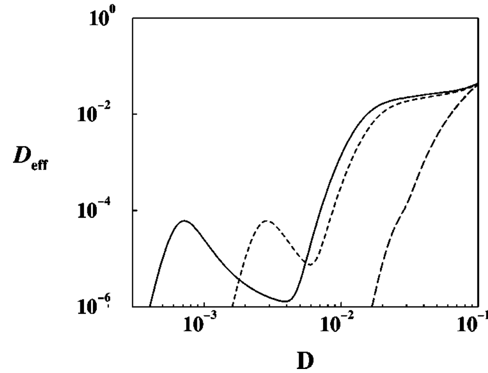

FIG. 7. $\mathcal{D}_{\rm eff}$ for $B=0$ (long dashed), $B=0.2$ (dashed), and $B=0.215$ (solid) of the dichotomic noise.

two-state system [9] with an effective barrier $\Delta U_{\rm eff}=\Delta U-B=0.05$. $\Delta U_{\rm eff}$ is nothing but the barrier related to the rate a_2 which, for large γ, is the dominating time scale for transitions to the input effected lower state. From the plot it can be seen that the behavior of the system with fast switching dichotomic noise, i.e., $\gamma\gg\Omega$, of large amplitude, i.e., $B\sim\Delta U$, is effectively equivalent to a periodically driven two-state system with reduced barrier height $\Delta U_{\rm eff}=\Delta U-B$. Note that a weak signal $A\ll\Delta U$ (LRT regime) in the absence of dichotomic noise can, for sufficiently large B, change its character to a strong signal, since $A\approx\Delta U_{\rm eff}$, in the presence of dichotomic noise.

For larger D a second locking region can be observed. In this region $\langle\omega_\sigma\rangle^s$ is synchronized with the dichotomic noise which is shown by the plateau around the value $\pi\gamma$. This locking is accompanied by a peak of the cross-correlation function $\langle\sigma\lambda\rangle^s$. Hence, in this region the output closely follows the dichotomic noise. The second locking region is less pronounced than the first one.

As shown in Fig. 6, for $\pi\gamma$ approaching Ω (from above) the two locking plateaus converge. When both frequencies match it is not clear in advance which signal the system should follow. Generally, one can state that for coinciding mean frequencies of the dichotomic noise $\pi\gamma$ and the periodic input Ω the system will be locked to the process with larger amplitude. For $\pi\gamma\ll\Omega$ the system again follows the slower signal (see the dot-dashed line in Fig. 6).

B. Phase locking

As stated above, the effect of frequency locking characterizes synchronization at the level of *average* motion of the input and output phases. It does not necessarily restrict fluctuations of the phase difference to small values.

Consequently, here we address the question whether frequency locking, detected in the system under consideration, is also accompanied by an effective phase locking. The measure we use to trace phase locking is the effective diffusion coefficient $\mathcal{D}_{\rm eff}$ for the phase difference φ already defined in Eq. (13). Three terms contribute to its general structure

$$\mathcal{D}_{\rm eff}=\mathcal{D}_d+\mathcal{D}_\sigma-\mathcal{D}_{\rm corr} \tag{23}$$

with $\mathcal{D}_d$ being the diffusion coefficient of the input and $\mathcal{D}_\sigma$ of the output. Cross correlations quantified by $\mathcal{D}_{\rm corr}=d/dt(\langle\phi_d\phi_\sigma\rangle-\langle\phi_d\rangle\langle\phi_\sigma\rangle)$ can decrease the effective diffusion coefficient. Since the deterministic periodic dynamics preserves any initial variance $\mathcal{D}_d$ vanishes.

The effective diffusion coefficient can be rewritten as

$$\mathcal{D}_{\rm eff}=\frac{\pi}{2}\langle\omega_\sigma\rangle+\pi[\langle\varphi W(\phi_\sigma,\phi_\lambda,\phi_d)\rangle-\langle\varphi\rangle \times\langle W(\phi_\sigma,\phi_\lambda,\phi_d)\rangle]. \tag{24}$$

Here, the dichotomic noise is hidden in the modified rate $W(\phi_\sigma,\phi_n,\phi_d)$ given by Eq. (15). By insertion of Eq. (15) into Eq. (24) one obtains

$$\frac{2}{\pi}\mathcal{D}_{\rm eff}=\langle\omega_\sigma\rangle-(a_1+a_2)\sinh\left(\frac{A}{D}\right)u_1-(a_2+a_1)\times\cosh\left(\frac{A}{D}\right)u_2+(a_2-a_1)\sinh\left(\frac{A}{D}\right)u_3, \tag{25}$$

where we have used the abbreviations

$$u_1=\langle(\varphi-\langle\varphi\rangle)\cos(\phi_\sigma-\phi_d)\rangle,$$

$$u_2=\langle(\varphi-\langle\varphi\rangle)\cos(\phi_\sigma-\phi_\lambda)\rangle,$$

$$u_3=\langle(\varphi-\langle\varphi\rangle)\cos(\phi_\lambda-\phi_d)\rangle. \tag{26}$$

Starting from the master equation (19), one has to derive equations for u_1, u_2, and u_3. This is a cumbersome but straightforward procedure which, after insertion of asymptotic stationary values, yields an explicit analytic expression for the effective diffusion coefficient.

In Fig. 7 the result is plotted for $\gamma=0.1$, $\Omega=0.001$, and different amplitudes of the local dichotomic process. The solid line corresponds to the same parameters that were used for the numerical simulation [see Fig. 2(b)]. By visual inspection the similarity is obvious. In the region of noise-induced frequency locking the diffusion coefficient also attains a minimum. Hence, frequency locking is accompanied by effective phase locking. In contrast, the second plateau of the output MSF at larger noise intensity is not shadowed by a second minimum of the diffusion coefficient. This, however, is explained by the fact that we have defined $\varphi = \phi_\sigma - \phi_d$ and $\mathcal{D}_{\mathrm{eff}}$ is not symmetric with respect to exchanging the roles of the periodic signal and dichotomic noise.

VI. SUMMARY AND CONCLUSIONS

We have investigated a periodically driven bistable system subjected to an additional dichotomic noise. An enhancement of noise-induced phase synchronization between the output and a slow but *weak* periodic input signal with increasing amplitude of the fast switching dichotomic noise was proved.

In general, we conclude that, for a slow and weak periodic input signal, by tuning the amplitude and rate of dichotomic noise we can control—enhance or suppress—the response of the system to the input. Optimal response is achieved for fast switching dichotomic noise with sufficiently large amplitude. This is obvious since in this case the dichotomic noise effectively reduces the original barrier ΔU by the amplitude B and changes a small amplitude signal (LRT) into a large amplitude signal (beyond LRT).

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